Determining Probabilities

Three methods for determining the probability of an occurrence:

- **Relative frequency method:** Repeat an experiment many times; assign as the probability the fraction $\frac{\text{occurrences}}{\#\text{ experiments run}}$.

  Example. Hit a bulls-eye 17 times out of 100; set the probability of hitting a bulls-eye to be $p(\text{bulls-eye}) = 0.17$.

- **Equal probability method:** Assume all outcomes have equal probability; assign as the probability $\frac{1}{\#\text{ of possible outcomes}}$.

  Example. Each side of a dodecahedral die is equally likely to appear; decide to set $p(1) = \frac{1}{12}$.

- **Subjective guess method:** If neither method above applies, give it your best guess.

  Example. How likely is it that your friend will come to a party?

Component Reliability

Many systems consist of components pieced together. To determine how reliable the system is, determine how reliable each component is and apply probability rules.

Definition: The **reliability** of a system is its probability of success.

Example. Launch the space shuttle into space with a three-stage rocket. $\boxed{\text{Stage 1}} \rightarrow \boxed{\text{Stage 2}} \rightarrow \boxed{\text{Stage 3}}$

⋆ In order for the rocket to launch, all components must succeed.

Let $R_1 = 90\%$, $R_2 = 95\%$, $R_3 = 96\%$ be the reliabilities of Stages 1–3.

$p(\text{system success}) = p(\text{S1 success and S2 success and S3 success})$

Component Reliability

Example. Communicating with the space shuttle.
There are two independent methods in which earth can communicate with the space shuttle

- A microwave radio with reliability $R_1 = 0.95$
- An FM radio, with reliability $R_2 = 0.96$.

$\star$ In order to be able to communicate with the shuttle, at least one of components must work.

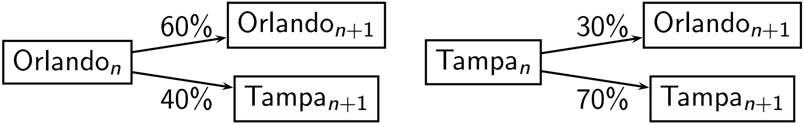
$p(\text{system success}) = p(\text{MW radio success or FM radio success})$

Markov Chains

A **Markov chain** is a sequence of random variables from some sample space, each corresponding to a successive time interval. From one time interval to the next, there is a *fixed* probability $a_{i,j}$ of transitioning from state j to state i. No transition depends on a past transition.

Keep track of these probabilities in an associated transition matrix A.

Example. Suppose you run a rental company based in Orlando and Tampa, Florida. People often drive between the cities; cars can be picked up and dropped off in either city. Suppose that historically,



What distribution of cars can the company expect in the long run?

Markov Chains

We will model this situation with a Markov Chain.

The historical data suggest that with a probability of 0.6, a car in Orlando at time n will be in Orlando at time $n+1$. Use this and the other expected transition probabilities to form the transition matrix A.

$$\textbf{TO:}\quad \begin{array}{c} \\ \text{Or} \\ \text{Tm} \end{array} \begin{array}{cc} \multicolumn{2}{c}{\textbf{FROM}:} \\ \text{Or} & \text{Tm} \end{array}$$

$$\begin{array}{c} \text{Or} \\ \text{Tm} \end{array} \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} = A,$$

- Let o_n be the probability that a car is in Orlando on day n
- Let t_n be the probability that a car is in Tampa on day n.

We can represent the distribution of cars at time n with the vector $\vec{\mathbf{x}}_n = \begin{bmatrix} o_n \\ t_n \end{bmatrix}$; notice that $\vec{\mathbf{x}}_{n+1} = \begin{bmatrix} o_{n+1} \\ t_{n+1} \end{bmatrix} = A \cdot \begin{bmatrix} o_n \\ t_n \end{bmatrix} = A\vec{\mathbf{x}}_n$ will be the distribution at time $n+1$. Given an initial distribution, $\vec{\mathbf{x}}_0 = \begin{bmatrix} o_0 \\ t_0 \end{bmatrix}$, the expected distribution of cars at time n is $\vec{\mathbf{x}}_n = A^n\vec{\mathbf{x}}_0$.

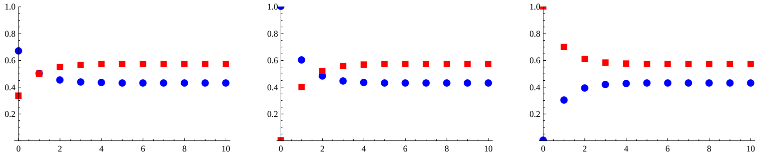
Markov Chains

For example, if they company starts off with twice as many cars in Orlando as in Tampa, then $\vec{\mathbf{x}}_0 = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$, so we expect

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \qquad \\ \qquad \end{bmatrix}.$$

$$\vec{\mathbf{x}}_2 = \begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} \qquad \\ \qquad \end{bmatrix} = \begin{bmatrix} \qquad \\ \qquad \end{bmatrix}.$$

How do we determine the expected distribution in the long run?



Markov Chains

Definition: Given a Markov Chain with transition matrix A, an **equilibrium distribution** is a vector $\vec{\mathbf{x}}_{eq}$ that satisfies $A\vec{\mathbf{x}}_{eq} = \vec{\mathbf{x}}_{eq}$.

[Linear Algebra: $\vec{\mathbf{x}}_{eq}$ *is an eigenvector corresponding to* $\lambda = 1$.]

In our example, the equilibrium distribution satisfies

$$\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix} = \begin{bmatrix} o_{eq} \\ t_{eq} \end{bmatrix}.$$

That is, $0.6o_{eq} + 0.3t_{eq} = o_{eq}$ and $0.4o_{eq} + 0.7t_{eq} = t_{eq}$.
Both equations reduce to $0.3t_{eq} = 0.4o_{eq}$, so $o_{eq} = \frac{3}{4}t_{eq}$.

Conclusion: If the company has 7000 cars in all, they would expect that in the long run, ____

Note on Markov Chains: The sum of the entries in every column of A is 1, because the total probability of transitioning **from** state i is 1. There is no rule for what the row sum will be.